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Spectral analysis in magnetohydrodynamic equilibria

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Abstract. It has been universally assumed that the spectrum of the magnetohydrodynamics equations, linearized around an equilibrium state, provides enough information on the short-term evolution of the plasma to study certain stability properties. We show that this is true if one takes into account viscous and resistive effects and the equilibrium satisfies certain regularity conditions.

1. Introduction

One of the oldest and mathematically more settled theories within magnetohydrodynamics is the linear stability theory of equilibrium states. The initial reason was practical: the best situation for the high-density, high-temperature plasmas needed in nuclear fusion would be a static one so that confinement would be guaranteed. To see for how long the plasma could be maintained in that state, one needs to analyse how it responds to small perturbations. It was confidently assumed that at least the initial evolution of the plasma would obey the linearized magnetohydrodynamics (MHD) equations, and that the classical stability analysis of the spectrum of this system would provide the necessary information (see, for example, [1] and references therein). For further simplification, since viscosity and resistivity are usually very low for those plasmas, they were taken as zero (the ideal MHD model: see [2]). The resulting system, although still extremely complicated, gives a wealth of results about several types of oscillation modes associated to different parts of the spectrum. The point we wish to make is that this ideal linearized approach gives only partial information on the real evolution of the plasma, even at an early stage; whereas the same analysis with positive (no matter how small) viscosity and resistivity provides a much more adequate view of both the linear and the nonlinear evolution, provided the equilibrium possesses a certain degree of regularity. The proofs are not always elementary and require some rather recent results on nonlinear evolution equations: for this reason we will restrict ourselves to the incompressible case, which is more amenable to this setting. In this situation the MHD equations are

$$\begin{aligned}
&\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p - (\operatorname{curl}\boldsymbol{B}\times\boldsymbol{B}) = \mathbf{0}\\
&\frac{\partial \boldsymbol{B}}{\partial t} - \operatorname{curl}(\boldsymbol{u}\times\boldsymbol{B}) - \eta\Delta\boldsymbol{B} = \mathbf{0}\\
&\operatorname{div}\boldsymbol{u} = 0\\
&\operatorname{div}\boldsymbol{B} = 0
\end{aligned} \tag{1}$$

where $\boldsymbol{u}$ is the fluid velocity, $\boldsymbol{B}$ the magnetic field, p the kinetic pressure, ν the (scalar) viscosity and η the resistivity: the density and constants present in the equations are normalized as 1. We have

$$\begin{aligned} &\operatorname{curl}\boldsymbol{B}\times\boldsymbol{B} = \boldsymbol{B}\cdot\nabla\boldsymbol{B} - \tfrac{1}{2}\nabla B^2 \\ &\operatorname{curl}(\boldsymbol{u}\times\boldsymbol{B}) = \boldsymbol{B}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{B}. \end{aligned} \tag{2}$$

Initial and boundary conditions must be added. To avoid unnecessary complications, we will deal only with two well-studied cases [3, 4]: when the plasma is bounded by a perfect conductor, which means that $\boldsymbol{u}$, $\boldsymbol{B}\cdot\boldsymbol{n}$ and $\operatorname{curl}\boldsymbol{B}\times\boldsymbol{n}$ vanish at the boundary of the domain Ω, and the space-periodic case, when all the magnitudes are space periodic in a certain box Ω and the averages of $\boldsymbol{u}$ and $\boldsymbol{B}$ vanish: $\iint_\Omega \boldsymbol{u} = \iint_\Omega \boldsymbol{B} = \boldsymbol{0}$. The total pressure $p + \frac{1}{2}B^2$ may be eliminated by projecting on the space of null divergence fields, which kills all the gradients: we are left with

$$\begin{aligned} &\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \boldsymbol{B}\cdot\nabla\boldsymbol{B} - \nu\Delta\boldsymbol{u} = \boldsymbol{0} \\ &\frac{\partial\boldsymbol{B}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{B} - \boldsymbol{B}\cdot\nabla\boldsymbol{u} - \eta\Delta\boldsymbol{B} = \boldsymbol{0} \\ &\operatorname{div}\boldsymbol{u} = 0 \\ &\operatorname{div}\boldsymbol{B} = 0. \end{aligned} \tag{3}$$

If we denote by $\boldsymbol{w}$ the six-dimensional vector $\boldsymbol{w} = (\boldsymbol{u}; \boldsymbol{B})$, and by $C(\boldsymbol{w}_1, \boldsymbol{w}_2)$ the vector

$$(\boldsymbol{u}_1\cdot\nabla\boldsymbol{u}_2 - \boldsymbol{B}_1\cdot\nabla\boldsymbol{B}_2; \boldsymbol{u}_1\cdot\nabla\boldsymbol{B}_2 - \boldsymbol{B}_1\cdot\nabla\boldsymbol{u}_2)$$

the system may be summarized as

$$\begin{aligned} &\frac{\partial\boldsymbol{w}}{\partial t} + A\boldsymbol{w} + C(\boldsymbol{w},\boldsymbol{w}) = \boldsymbol{0} \\ &\boldsymbol{w}(0) = \boldsymbol{w}_0 \end{aligned} \tag{4}$$

where A is the elliptic operator $(-\nu\Delta; -\eta\Delta)$. The main spaces are, in the perfect conductor case,

$$\begin{aligned} &H = \{\boldsymbol{w} = (\boldsymbol{u}; \boldsymbol{B}) \in L^2(\Omega)^6 : \operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{B} = 0, \boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega} = \boldsymbol{B}\cdot\boldsymbol{n}|_{\partial\Omega} = 0\} \\ &V = \{\boldsymbol{w} = (\boldsymbol{u}; \boldsymbol{B}) \in H_0^1(\Omega)^3 \times H^1(\Omega)^3 : \operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{B} = 0, \boldsymbol{B}\cdot\boldsymbol{n}|_{\partial\Omega} = 0\}. \end{aligned} \tag{5}$$

In the periodic case,

$$\begin{aligned} &H = \left\{\boldsymbol{w} = (\boldsymbol{u}; \boldsymbol{B}) \in L^2(\Omega)^6 : \boldsymbol{w}\text{ periodic, } \operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{B} = 0, \iint_\Omega \boldsymbol{u} = \iint_\Omega \boldsymbol{B} = \boldsymbol{0}\right\} \\ &V = H \cap H^1(\Omega)^6. \end{aligned} \tag{6}$$

C takes $V \times V$ into the dual space V', satisfying

$$(C(\boldsymbol{w}_1,\boldsymbol{w}_2),\boldsymbol{w}_3) = -(C(\boldsymbol{w}_1,\boldsymbol{w}_3),\boldsymbol{w}_2) \tag{7}$$

which implies

$$(C(\boldsymbol{w}_1,\boldsymbol{w}_2),\boldsymbol{w}_2) = 0. \tag{8}$$

The L^2-norm in H will be denoted by $|\ |$, and the H^1-norm in V by $\|\ \|$.

2. Ideal MHD equilibria

Take in the MHD system $\eta = \nu = 0$, and consider a stationary solution of it ($\partial/\partial t = 0$). The very existence of this solution is problematic: for configurations with a certain degree of symmetry, in which the magnitudes depend on less than the three space variable, solutions may be found. In particular, axisymmetric equilibria satisfy the well known Grad–Shafranov equation [1]. Grad even suggested that non-pathological totally asymmetric solutions could not exist [5], although later weak asymmetric equilibria (with discontinuities in the field) were found [6, 7]. Anyway, assume that we have an ideal stationary solution $\boldsymbol{w}_0$. The ideal MHD equations linearized around $\boldsymbol{w}_0$ are

$$\begin{aligned} &\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{v} + \nabla p_1 - (\operatorname{curl}\boldsymbol{B}\times\boldsymbol{b}) - (\operatorname{curl}\boldsymbol{b}\times\boldsymbol{B}) = \boldsymbol{0} \\ &\frac{\partial \boldsymbol{b}}{\partial t} - \operatorname{curl}(\boldsymbol{u}\times\boldsymbol{b}) - \operatorname{curl}(\boldsymbol{v}\times\boldsymbol{B}) = \boldsymbol{0} \\ &\operatorname{div}\boldsymbol{v} = 0 \\ &\operatorname{div}\boldsymbol{b} = 0 \end{aligned} \tag{9}$$

where the stationary solution is $(\boldsymbol{u}, \boldsymbol{B})$ and the perturbed quantities are $\boldsymbol{v}$ (velocity), $\boldsymbol{b}$ (field) and p_1 (pressure). In addition they must satisfy the same boundary conditions as the original magnitudes. Let us write this system as

$$\frac{\partial \boldsymbol{w}}{\partial t} = L\boldsymbol{w}. \tag{10}$$

To study its solution, the first thing to ask is whether L is the infinitesimal generator of a semigroup. To prove that this is really so is not simple at all: a rigorous proof, using the theorem of Hille–Yosida for $L + \lambda$ with several natural definitions of the domain $D(L)$ is shown in [8]. One of those domains is formed by solenoidal functions with the boundary conditions $\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$, $\boldsymbol{b}\cdot\boldsymbol{n}|_{\partial\Omega} = 0$. It is also well known [1] that for a static equilibrium ($\boldsymbol{u} = \boldsymbol{0}$), the system above may be written as a second-order equation

$$\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = F\boldsymbol{\xi} \tag{11}$$

where ξ is the fluid displacement and F is a symmetric operator for several boundary conditions. Apparently this solves many problems related to semigroup existence and spectral theorems, but still one needs to study the existence of self-adjoint extensions of $F + \lambda$ for some λ [8], and worse, the spectrum of F includes all the $\omega^2 : \omega \in \sigma(L)$, but not vice versa [5]. Thus its study may provide only necessary conditions for stability, such as the negativity of F ($(F\boldsymbol{\xi}, \boldsymbol{\xi}) \leqslant 0$ for admissible $\boldsymbol{\xi}$).

Returning to the original first-order equation, even if L generates a semigroup S, we only have (see, for example, [9])

$$\mathrm{e}^{t\,\sigma(L)} \subset \sigma(S(t)) \tag{12}$$

so that if $\sigma(L) \cap \{\operatorname{Re} z > 0\} \neq \emptyset$, there exist exponentially growing modes and therefore instability, but $\sigma(L) \subset \{\operatorname{Re} z \leqslant 0\}$ does not guarantee any kind of stability. When L or $L + \lambda$ is self-adjoint the spectral theorem gives the identity of these sets and relates the norm of $S(t)$ to the location of the spectrum, but this is not the case.

We see that the ideal spectrum simply does not give enough information to foresee even the linear evolution of the plasma; this does not encourage any effort to compare the linear and the nonlinear behaviour. Things are different when a positive viscosity and resistivity are introduced. Although these may be extremely small, the spectrum is essentially different and much more meaningful.

3. Resistive MHD equilibria

As mentioned previously, the incompressible MHD system may be written as

$$\begin{aligned}&\frac{\partial \boldsymbol{w}}{\partial t} + A\boldsymbol{w} + C(\boldsymbol{w}, \boldsymbol{w}) = \boldsymbol{0}\\&\boldsymbol{w}(0) = \boldsymbol{w}_0\end{aligned} \tag{13}$$

so that any equilibrium state $\boldsymbol{w}_0$ must satisfy

$$A\boldsymbol{w}_0 + C(\boldsymbol{w}_0, \boldsymbol{w}_0) = \boldsymbol{0} \tag{14}$$

which is a nonlinear elliptic system; theorems proving existence and regularity of solutions are available (see, for example, [10, 11]). The linearized equations around $\boldsymbol{w}_0$ are

$$\begin{aligned}&\frac{\partial \boldsymbol{w}}{\partial t} + A\boldsymbol{w} + C(\boldsymbol{w}, \boldsymbol{w}_0) + C(\boldsymbol{w}_0, \boldsymbol{w}) = \boldsymbol{0}\\&\boldsymbol{w}(0) = \boldsymbol{w}_1\end{aligned} \tag{15}$$

which we will write, as before, by

$$\begin{aligned}&\frac{\partial \boldsymbol{w}}{\partial t} = L\boldsymbol{w}\\&\boldsymbol{w}(0) = \boldsymbol{w}_1.\end{aligned} \tag{16}$$

Lemma 1. If $\boldsymbol{w}_0 \in L^\infty(\Omega)$, there exists positive k and α such that

$$((-L + k)\, \boldsymbol{w}, \boldsymbol{w}) \geqslant \alpha \|\boldsymbol{w}\|^2 \tag{17}$$

for all $\boldsymbol{w} \in V$.

Proof. Since $(C(\boldsymbol{w}_0, \boldsymbol{w}), \boldsymbol{w}) = 0$, $(C(\boldsymbol{w}, \boldsymbol{w}), \boldsymbol{w}_0) = -(C(\boldsymbol{w}, \boldsymbol{w}_0), \boldsymbol{w})$, it is enough to analyse summands of the form $(\boldsymbol{f} \cdot \nabla \boldsymbol{g})\boldsymbol{h}_0$, where $\boldsymbol{f}$ and $\boldsymbol{g}$ are part of $\boldsymbol{w}$, $\boldsymbol{h}_0$ part of $\boldsymbol{w}_0$. We have

$$|(\boldsymbol{f} \cdot \nabla \boldsymbol{g})\, \boldsymbol{h}_0| \leqslant |\boldsymbol{f}|\, \|\boldsymbol{g}\|\, \|\boldsymbol{h}_0\|_\infty \leqslant |\boldsymbol{w}|\, \|\boldsymbol{w}\|\, \|\boldsymbol{w}_0\|_\infty. \tag{18}$$

Let $\delta = \inf\{\nu, \eta\}$. If we take k such that $2\|\boldsymbol{w}_0\|_\infty \leqslant \sqrt{k\delta}$,

$$\delta \|\boldsymbol{w}\|^2 - |\boldsymbol{w}|\, \|\boldsymbol{w}\|\, \|\boldsymbol{w}_0\|_\infty + k|\boldsymbol{w}|^2 \geqslant \alpha \|\boldsymbol{w}\|^2 \tag{19}$$

for some $\alpha > 0$. Since $(A\boldsymbol{w}, \boldsymbol{w}) \geqslant \delta \|\boldsymbol{w}\|^2$, the result is proved. □

Notice that since Ω is a three-dimensional domain, if $s > \frac{3}{2}$, $H^s(\Omega) \subset \mathcal{C}(\bar{\Omega})$. Thus if $\boldsymbol{w}_0 \in H^s(\Omega)$, the result is true.

Proposition 2. There exists a unique solution of system (15). Moreover, the mapping $\boldsymbol{w}_1 \to \boldsymbol{w}(t)$ is continuous from H to $H^2(\Omega)^6$.

Proof. Since $-L + k$ is a linear elliptic accretive operator, there exists a unique solution of

$$\begin{aligned}&\frac{\partial \boldsymbol{w}}{\partial t} = L\boldsymbol{w} - k\boldsymbol{w}\\&\boldsymbol{w}(0) = \boldsymbol{w}_1\end{aligned} \tag{20}$$

and $\boldsymbol{w}_1 \to \boldsymbol{w}(t)$ takes H to $D(A) \subset H^2(\Omega)^6$ (see, for example, [4]). The solution to our system is this one multiplied by e^{kt}, which satisfies the same regularity property for a fixed t. □

Let $S(t)$ be the semigroup generated by (15).

Proposition 3.

$$\sigma(S(t)) = \mathrm{e}^{t\sigma(L)} \cup \{0\}. \tag{21}$$

Proof. In addition to the general inclusion (12), we also have [9]

$$\sigma_p(S(t)) \subset \mathrm{e}^{t\sigma_p(L)} \cup \{0\} \tag{22}$$

where σ_p denotes the point spectrum. Since the inclusion $H^2(\Omega)^6 \subset H$ (and even $H^2(\Omega)^6 \subset V$) are compact, $S(t)$ is a compact operator, so the whole of its spectrum except perhaps zero, is included in $\sigma_p(S(t))$. Zero always belongs to $\sigma(S(t))$. This value could even be taken as the exponential of the value $-\infty$, which is a limit spectral point of the elliptic operator L. Notice also that the spectrum of L is formed only by eigenvalues. □

Hence the spectrum of $S(t)$ is indeed determined by $\sigma(L)$: any exponentially growing solution must correspond to an eigenvalue of L, and vice versa. This is still not enough: what we need is to know the behaviour of $S(t)\,\boldsymbol{w}_1$ for any initial condition $\boldsymbol{w}_1$ near the equilibrium $\boldsymbol{w}_0$, not just for the eigenfunctions. If $S(t)$ is self-adjoint, this follows from the spectral decomposition theorem. Since it is not, it remains to be seen whether its spectrum determines somewhat the size of the solutions as they evolve in time. This is true because the operators $S(t)$ are compact and hence the semigroup is *almost-stable* [12, 13]. This means the following: let $\rho > 0$ be a regular radius, i.e. a radius such that there is no eigenvalue of $S(1)$ with modulus ρ. Let $\sigma_+(\rho) = \sigma(S(1)) \cap \{|z| > \rho\}$, $\sigma_-(\rho) = \sigma(S(1)) \cap \{|z| < \rho\}$, $H_+(\rho)$ the (finite-dimensional) invariant subspace of $S(1)$ associated to $\sigma_+(\rho)$, $H_-(\rho)$ the invariant subspace corresponding to $\sigma_-(\rho)$. We have $H = H_+(\rho) \oplus H_-(\rho)$, although the decomposition does not need to be orthogonal. Then there exist norms $|\ |_+$ in $H_+(\rho)$, $|\ |_-$ in $H_-(\rho)$ such that the norm $|\ |$ in H is equivalent to $\sup\{|\ |_+, |\ |_-\}$, in the sense that if we denote by $P_\pm$ the projection into $H_\pm(\rho)$, the norms $|\boldsymbol{w}|$ and $\sup\{|P_+\boldsymbol{w}|_+, |P_-\boldsymbol{w}|_-\}$ are equivalent. Also the restrictions of $S(t)$ to the invariant subspaces $H_\pm(\rho)$ satisfy

$$\|S(t)|_{H_-(\rho)}\|_- \leqslant (\rho - \varepsilon)^t \tag{23}$$

$$\left\|(S(t)|_{H_+(\rho)})^{-1}\right\|_+ \leqslant (\rho + \varepsilon)^{-t} \tag{24}$$

for some $\varepsilon > 0$.

Hence the states within $H_-(\rho)$ tend to $\boldsymbol{w}_0$ at least as rapidly as $(\rho - \varepsilon)^t$: thus for any initial condition $\boldsymbol{w}_1$,

$$|S(t)\,\boldsymbol{w}_1 - P_+(S(t)\,\boldsymbol{w}_1)| = |P_-(S(t)\,\boldsymbol{w}_1)| \leqslant C(\rho - \varepsilon)^t |\boldsymbol{w}_1| \tag{25}$$

for some fixed constant C. The trajectory $P_+S(t)\,\boldsymbol{w}_1 = S(t)P_+\boldsymbol{w}_1$ lies within the finite-dimensional space $H_+(\rho)$. Therefore any trajectory may be approximated by one associated to eigenvalues larger than ρ, i.e. eigenvalues of L whose real part is larger than $\log\rho$, which hence determine the stability of the system. If 1 is such a radius and $H_+(1) = 0$, the equilibrium is linearly stable: all the nearby trajectories tend to fall into $\boldsymbol{w}_0$. If $H_+(1) \neq 0$, the equilibrium is unstable. The doubtful case occurs when 1 is not a regular radius, i.e. when there are eigenvalues of L of real part 0. Apparently this is unfortunate, because several portions of the ideal spectrum are purely imaginary in many important cases. Let us remember, however, that there is no guarantee that the spectrum of L for nonideal conditions must approach the ideal spectrum when viscosity and resistivity tend to zero. This is a singular perturbation problem, for which there are not general theorems: indeed, there are examples [14] showing that (i) it may happen that a regular radius for the ideal spectrum is not so for the resistive spectrum, no matter how small ν and η; and (ii) ρ (in particular 1) may be a regular radius for every resistive spectrum, and nevertheless even the whole ideal spectrum may be contained within $\{\mathrm{Re}\ z = \log \rho\}$. Hence this ideal spectrum provides few clues on the resistive evolution.

4. Nonlinear evolution near equilibria

We now reach the real problem: obviously we cannot expect for the solutions of the linearized system to mimic the real ones for long periods of time, but is it true that they do resemble them over the short term? This is by no means obvious. We will prove that such similarity exists if (i) $\boldsymbol{w}_0 \in L^\infty(\Omega) \cap V$ and (ii) the initial conditions to be studied, besides being near $\boldsymbol{w}_0$ in the H-norm, belong to V (although they do not need to approach $\boldsymbol{w}_0$ in the V-norm).

Let us begin by remembering some facts about the MHD incompressible equations [4]. In dimension 3, there is so far no proof of existence and uniqueness; there exist weak bounded solutions, and for any initial condition $\boldsymbol{w}_1 \in V$, there exists a finite time T_1 depending only on $\|\boldsymbol{w}_1\|$ such that a unique solution $\boldsymbol{w}$ of the problem exists for $t \in [0, T_1]$; also

$$\boldsymbol{w} \in L^2\big([0, T_1], H^2(\Omega)^6\big) \cap \mathcal{C}([0, T_1], V). \tag{26}$$

If we take $\|\boldsymbol{w}_1\| \leqslant R$, $T_1(R)$ may be taken such that $\|\boldsymbol{w}(t)\| \leqslant 2(1 + R)$, $\forall t \in [0, T_1(R)]$. Thus, although in principle any solution may blow up or bifurcate, it remains smooth at least for a fixed time depending only on $\|\boldsymbol{w}(0)\|$. If we define $\mathcal{D}_\rho(T)$ as the set of initial conditions $\boldsymbol{w}_1 \in V$ such that the solution $\boldsymbol{w}(t)$ exists at least within $[0, T]$ and $\|\boldsymbol{w}(t)\| \leqslant \rho\ \forall t \in [0, T]$, it follows that $\bar{B}_V(0, R) \subset \mathcal{D}_\rho(T_1(R))$, with $\rho = 2(1 + R)$.

Theorem 4. Let $\boldsymbol{w}_0$ be an equilibrium of the MHD equations, and assume $\boldsymbol{w}_0 \in L^\infty(\Omega) \cap V$. Let $\boldsymbol{w}(t)$ denote the solution of the nonlinear system with initial condition $\boldsymbol{w}_0 + \boldsymbol{z}$, where $\boldsymbol{z} \in V$ and $\|\boldsymbol{w}_0 + \boldsymbol{z}\| \leqslant R$; let $\boldsymbol{w}^*(t)$ be the solution of the equations of the MHD linearized at $\boldsymbol{w}_0$ with initial condition $\boldsymbol{w}^*(0) = \boldsymbol{z}$. Then

$$\boldsymbol{w}(t) - \boldsymbol{w}_0 = \boldsymbol{w}^*(t) + E(t)\,\boldsymbol{z} \tag{27}$$

where $|E(t)\,\boldsymbol{z}|/|\boldsymbol{z}| \to 0$ when $|\boldsymbol{z}| \to 0$, uniformly in $t \in [0, T_1(R)]$.

Proof. Let $T(t)\,\boldsymbol{w}_1$ denote the solution of the nonlinear problem with initial condition $\boldsymbol{w}_1$. $T(t)$ is a semigroup up to the time $T_1(R)$ for all $\boldsymbol{w}_1 \in \bar{B}_V(0, R)$. Then $\mathcal{D}_\rho(T_1(R))$ is open in V, and $T(t)$ Fréchet differentiable at every point of it, with the H-norm (see [15]). Moreover its differential at a point $\boldsymbol{w}_1$, $T'(t)(\boldsymbol{w}_1)$, satisfies that its action on $\boldsymbol{z} \in V$ is the value taken at time t by the solution of the linearized equation

$$\begin{aligned} &\frac{\partial \boldsymbol{z}}{\partial t} + A\boldsymbol{z} + C(\boldsymbol{z}, \boldsymbol{w}_1) + C(\boldsymbol{w}_1, \boldsymbol{z}) = \boldsymbol{0} \\ &\boldsymbol{z}(0) = \boldsymbol{z}. \end{aligned} \tag{28}$$

By the definition of Fréchet differentiability, this means

$$T(t)(\boldsymbol{w}_1 + \boldsymbol{z}) = T(t)(\boldsymbol{w}_1) + T'(t)(\boldsymbol{w}_1)\,\boldsymbol{z} + E(t)\,\boldsymbol{z} \tag{29}$$

and $|E(t)\,\boldsymbol{z}|/|\boldsymbol{z}| \to 0$ when $|\boldsymbol{z}| \to 0$, uniformly for $t \in [0, T_1(R)]$.

If we set $\boldsymbol{w}_1 = \boldsymbol{w}_0$, $T(t)\,\boldsymbol{w}_0 = \boldsymbol{w}_0$ for all t and $T'(t)(\boldsymbol{w}_0)$ coincides with the linear semigroup $S(t)$ of the previous section, i.e. $S(t)\,\boldsymbol{z} = \boldsymbol{w}^*(t)$, which proves the theorem. □

Corollary 5. Let ρ be a regular radius of the linearized equation. For any initial condition $\boldsymbol{w}_1 \in V$ such that $\|\boldsymbol{w}_1\| \leqslant R$, there exists a trajectory of the linearized system $\boldsymbol{w}^*(t)$, contained in the finite-dimensional space $H_+(\rho)$, such that

$$|\boldsymbol{w}(t) - \boldsymbol{w}^*(t)| \leqslant C\rho^t + |E(t)(\boldsymbol{w}_1 - \boldsymbol{w}_0)| \tag{30}$$

where $|E(t)(\boldsymbol{w}_1 - \boldsymbol{w}_0)|/|\boldsymbol{w}_1 - \boldsymbol{w}_0| \to 0$ as $|\boldsymbol{w}_1 - \boldsymbol{w}_0| \to 0$, uniformly in $t \in [0, T_1(R)]$.

Proof. It is an obvious consequence of theorem 4 and proposition 3. Hence nonlinear stability is determined by $\boldsymbol{w}^*(t)$, i.e. by the linear modes associated to spectral points larger than ρ, at least up to $T_1(R)$. □

This result may be refined in a number of ways: the trajectory $\boldsymbol{w}^*(t)$ may be changed by another one staying within a finite-dimensional invariant manifold, tangent to $H_+(\rho)$, instead of within the linear space itself, and the error term $E(t)$ omitted; this adds little, however, to the approximation itself. Also a related theorem may be proved in $H^s(\Omega)$, for $s > \frac{3}{2}$. [13, 16]. In this case $\|E(t)\,\boldsymbol{z}\|_s/\|\boldsymbol{z}\|_s \to 0$ when $\|\boldsymbol{z}\|_s \to 0$, which is a stronger approximation; it is attained, however, at the prize that the initial condition must be near $\boldsymbol{w}_0$ not only in the H-norm, i.e. in energy, but in the finer norm of $H^s(\Omega)$. Energy is a more natural way of measuring perturbation sizes.

As a conclusion, the spectrum of the viscous, resistive linearized MHD equations at an equilibrium provides fairly satisfactory information on the initial evolution of the system and therefore of its short-term stability. This cannot be said of ideal MHD, where even the linear evolution is much less controlled by the spectrum.

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